1. Find the value of

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$$\sum_{r=1}^{200} (r+1)(r-1)$$

(4)

$$\sum_{1}^{n} r^2 - 1 = \sum_{1}^{n} r^2 - \sum_{1}^{n} 1 = \frac{1}{6}n(n+1)(2n+1) - n = \frac{1}{6}n\left[2n^2+3n+1-6\right]$$

$$= \frac{1}{6}n\left[2n^2+3n-5\right] = \frac{1}{6}n(n-1)(2n+5)$$

$$\therefore \sum_{1}^{200} r^2 - 1 = \frac{1}{6}(200)(199)(405) = 2686500$$

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2. Given that $-2 + 3i$ is a root of the equation

$$z^2 + pz + q = 0$$

where p and q are real constants,

(a) write down the other root of the equation.

(1)

(b) Find the value of p and the value of q.

(3)

$$z = -2+3i \Rightarrow z^* = -2-3i$$

$$(z+2-3i)(z+2+3i) = 0$$

$$z^2 + 4z + (4+9) = 0 \Rightarrow z^2 + 4z + 13 = 0$$

3.

$$\mathbf{A} = \begin{pmatrix} 4 & -2 \\ a & -3 \end{pmatrix}$$

where a is a real constant and $a \neq 6$

(a) Find $\mathbf{A}^{-1}$ in terms of a.

(3)

Given that $\mathbf{A} + 2\mathbf{A}^{-1} = \mathbf{I}$, where $\mathbf{I}$ is the 2×2 identity matrix,

(b) find the value of a.

(3)

a) $A^{-1} = \frac{1}{\det A}\begin{pmatrix} -3 & 2 \\ -a & 4 \end{pmatrix}$ $\quad \det A = -12 + 2a = 2(a-6)$

$\therefore A^{-1} = \frac{1}{2(a-6)}\begin{pmatrix} -3 & 2 \\ -a & 4 \end{pmatrix}$

b) $\begin{pmatrix} 4 & -2 \\ a & -3 \end{pmatrix} + \frac{1}{a-6}\begin{pmatrix} -3 & 2 \\ -a & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$-2 + \frac{2}{a-6} = 0 \quad \therefore \quad \frac{2}{a-6} = 2 \quad \therefore a-6=1$

$\therefore a = 7$

4. $$\mathrm{f}(x) = x^{\frac{3}{2}} - 3x^{\frac{1}{2}} - 3, \qquad x > 0$$

Given that α is the only real root of the equation $\mathrm{f}(x) = 0$,

(a) show that $4 < \alpha < 5$

(2)

(b) Taking 4.5 as a first approximation to α, apply the Newton-Raphson procedure once to $\mathrm{f}(x)$ to find a second approximation to α, giving your answer to 3 decimal places.

(5)

(c) Use linear interpolation once on the interval [4, 5] to find another approximation to α, giving your answer to 3 decimal places.

(3)

a) $f(4) = 4^{\frac{3}{2}} - 3x^{\frac{1}{2}} - 3 = (8) - 3(2) - 3 = -1 < 0$

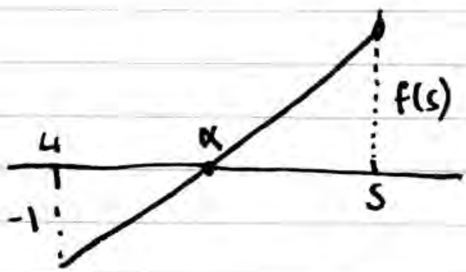
$f(5) = \sqrt{125} - 3\sqrt{5} - 3 = 2\sqrt{5} - 3 = 1.47 > 0$

∴ by sign change rule $\alpha \in (4,5)$

b) $\alpha = 4.5 - \dfrac{(4.5)}{f'(4.5)} = 4.426$

$f'(x) = \frac{3}{2}x^{\frac{1}{2}} - \frac{3}{2}x^{-\frac{1}{2}}$

c)



$\dfrac{5-\alpha}{f(5)} = \dfrac{\alpha - 4}{1}$

$5 - \alpha = 1.4721...\alpha - 5.888...$

$10.8885... = 2.4721...\alpha$

$\alpha = 4.405$

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5. Given that $z_1 = -3 - 4i$ and $z_2 = 4 - 3i$

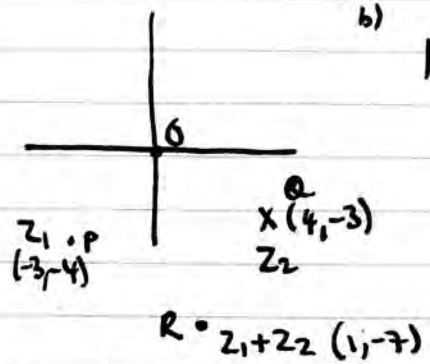
(a) show, on an Argand diagram, the point P representing z_1 and the point Q representing z_2 **(2)**

(b) Given that O is the origin, show that OP is perpendicular to OQ. **(2)**

(c) Show the point R on your diagram, where R represents $z_1 + z_2$ **(1)**

(d) Prove that $OPRQ$ is a square. **(2)**

a)



b) $M_{OP} = \frac{4}{3}$ $\quad M_{OQ} = \frac{-3}{4}$

$M_{OP} \times M_{OQ} = -1 \quad \therefore$ perp

c) $z_1 + z_2 = 1 - 7i$

d) $M_{PR} = \frac{-3}{4}$ $\therefore$ Perp to OP and Parallel to OQ

$|OP| = |PR| = |RQ| = |OQ| = \sqrt{3^2+4^2} = 5$

$\therefore$ 4 sides equal in length

$\therefore$ 4 interior angles must be $90°$

$\therefore$ Square.

6. It is given that α and β are roots of the equation $3x^2 + 5x - 1 = 0$

(a) Find the exact value of $\alpha^3 + \beta^3$

(3)

(b) Find a quadratic equation which has roots $\frac{\alpha^2}{\beta}$ and $\frac{\beta^2}{\alpha}$, giving your answer in the form $ax^2 + bx + c = 0$, where a, b and c are integers.

(5)

$x^2 + \frac{5}{3}x - \frac{1}{3} = 0 \Rightarrow \left(x + \frac{5}{6}\right)^2 - \frac{25}{36} - \frac{12}{36} = 0$

$\Rightarrow \left(x + \frac{5}{6}\right) = \frac{\pm\sqrt{37}}{6} \qquad x = -\frac{5}{6} \pm \frac{\sqrt{37}}{6}$

$6\alpha = -5 + \sqrt{37} = \sqrt{37} - 5 \qquad 6\beta = \sqrt{37} + 5$

$216\alpha^3 = (\sqrt{37})^3 - 3(37)(5) + 3\sqrt{37}(25) - 125$

$216\beta^3 = (\sqrt{37})^3 + 3(37)(5) + 3\sqrt{37}(25) + 125$ +

$216(\alpha^3 + \beta^3) = 2(\sqrt{37})^3 + 150\sqrt{37}$

$\therefore \alpha^3 + \beta^3 = \frac{2\sqrt{37}\left[37 + 75\right)}{216} = \frac{28\sqrt{37}}{27}$

$x^2 + \frac{5}{3}x - \frac{1}{3} = 0 \qquad x^2 + (\alpha + \beta)x + (\alpha\beta) = 0$

$\alpha + \beta = \frac{5}{3} \Rightarrow 3\alpha + 3\beta = 5 \qquad \alpha\beta = -\frac{1}{3}$

$(\alpha + \beta)^3 = (\alpha + \beta)(\alpha^2 + \beta^2 - \alpha\beta)$

$\left(x - \frac{\alpha^2}{\beta}\right)\left(x - \frac{\beta^2}{\alpha}\right)$

6. It is given that α and β are roots of the equation $3x^2 + 5x - 1 = 0$

(a) Find the exact value of $\alpha^3 + \beta^3$

(3)

(b) Find a quadratic equation which has roots $\frac{\alpha^2}{\beta}$ and $\frac{\beta^2}{\alpha}$, giving your answer in the form $ax^2 + bx + c = 0$, where a, b and c are integers.

(5)

$$3x^2+5x-1=0 \Rightarrow x^2+\frac{5}{3}x-\frac{1}{3}=0$$

$$x^2-(\alpha+\beta)x-(\alpha\beta)=0$$

$\therefore (\alpha+\beta)=\frac{-5}{3}$

$(\alpha+\beta)^2=\frac{25}{9}$

$\alpha\beta=-\frac{1}{3}$

$$(\alpha+\beta)^3=(\alpha+\beta)(\alpha^2+\beta^2-\alpha\beta)$$

$$=(\alpha+\beta)((\alpha+\beta)^2-3\alpha\beta)$$

$$=\left(\frac{-5}{3}\right)\left(\frac{25}{9}-3\left(-\frac{1}{3}\right)\right)$$

$$=\frac{-5}{3}\left(\frac{34}{9}\right)=\frac{-170}{27}$$

alt

$$\left(x+\frac{5}{6}\right)^2-\frac{25}{36}-\frac{12}{36} \Rightarrow x=\frac{-5\pm\sqrt{37}}{6}$$

$6\alpha=-5+\sqrt{37} \qquad 6\beta=-5-\sqrt{37}$

$$\Rightarrow 216\alpha^3=(-5+\sqrt{37})^3=(\sqrt{37}-5)^3=37\sqrt{37}-3(37)(5)+3(\sqrt{37})(25)-125$$

$$+\ 216\beta^3=(-5-\sqrt{37})^3=(-(\sqrt{37})-5)^3=-37\sqrt{37}-3(37)(5)-3(\sqrt{37})(25)-125\ +$$

$$\therefore 216(\alpha^3+\beta^3)=-1110-250 \qquad \therefore \alpha^3+\beta^3=\frac{-170}{27}.$$

b) $$x^2-\left(\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}\right)x+\frac{(\alpha\beta)^2}{\alpha\beta}=0$$

$$x^2-\left(\frac{\alpha^3+\beta^3}{\alpha\beta}\right)x+(\alpha\beta)=0$$

$$x^2-\left(\frac{\frac{-170}{27}}{-\frac{1}{3}}\right)x-\frac{1}{3}=0 \Rightarrow x^2-\frac{170}{9}x-\frac{1}{3}=0$$

$\times 9$

$$\therefore 9x^2-170x-3=0$$

7.

$$\mathbf{P} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

(a) Describe fully the single geometrical transformation U represented by the matrix $\mathbf{P}$.
(3)

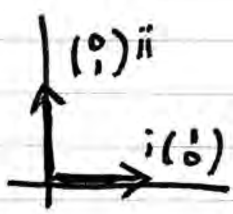
The transformation V, represented by the 2×2 matrix $\mathbf{Q}$, is a reflection in the x-axis.

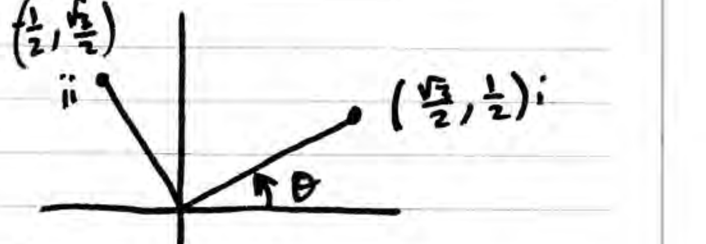
(b) Write down the matrix $\mathbf{Q}$.
(1)

Given that V followed by U is the transformation T, which is represented by the matrix $\mathbf{R}$,

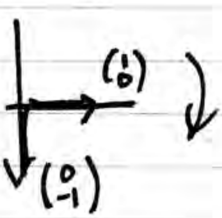
(c) find the matrix $\mathbf{R}$.
(2)

(d) Show that there is a real number k for which the transformation T maps the point $(1, k)$ onto itself. Give the exact value of k in its simplest form.
(5)

a) 

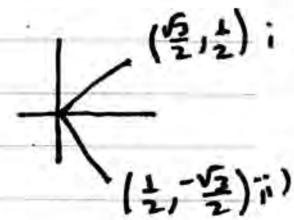
$\Rightarrow$ 

$\theta = \tan^{-1}\left(\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$ $\therefore$ anticlockwise rotation about O, by $\frac{\pi}{6}$.

b) 

$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

c) $R = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} +\frac{\sqrt{3}}{2} & +\frac{1}{2} \\ +\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$



d) $\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 1 \\ k \end{pmatrix} = \begin{pmatrix} 1 \\ k \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{\sqrt{3}}{2} + \frac{1}{2}k \\ \frac{1}{2} - \frac{\sqrt{3}}{2}k \end{pmatrix} = \begin{pmatrix} 1 \\ k \end{pmatrix} \Rightarrow \frac{1}{2}k = 1 - \frac{\sqrt{3}}{2} \therefore k = 2 - \sqrt{3}$

8. The hyperbola H has cartesian equation $xy = 16$
The parabola P has parametric equations $x = 8t^2$, $y = 16t$.

(a) Find, using algebra, the coordinates of the point A where H meets P.
(3)

Another point $B\,(8, 2)$ lies on the hyperbola H.

(b) Find the equation of the normal to H at the point $(8, 2)$, giving your answer in the form $y = mx + c$, where m and c are constants.
(5)

(c) Find the coordinates of the points where this normal at B meets the parabola P.
(6)

a) $xy = 16 \quad xy = 16(8t^3) \quad \therefore \quad 16 = 16(8t^3)$

$\therefore 8t^3 = 1 \quad \therefore \quad t^3 = \frac{1}{8} \quad \therefore \quad t = \frac{1}{2} \quad \Rightarrow x = 2 \quad y = 8$

b) $y = \frac{16}{x} = 16x^{-1} \quad \frac{du}{dx} = -16x^{-2} = -\frac{16}{x^2}$

at $(8,2)$ $M_t = \frac{-16}{64} = -\frac{1}{4} \quad \therefore M_n = 4$

$y - 2 = 4(x-8) \Rightarrow y - 2 = 4x - 32 \Rightarrow y = 4x - 30$

c) $y = 4x - 30 \Rightarrow 16t = 32t^2 - 30$

$\Rightarrow 32t^2 - 16t - 30 = 0 \quad \Rightarrow 16t^2 - 8t - 15 = 0$

$(4t-5)(4t+3) = 0$

$t = \frac{5}{4} \quad t = -\frac{3}{4}$

$t = \frac{5}{4} \quad x = 8t^2 = \frac{25}{2} \quad y = 16t = 20 \quad \left(\frac{25}{2}, 20\right)$

$t = -\frac{3}{4} \quad x = \frac{9}{2} \quad y = -12 \quad \left(\frac{9}{2}, -12\right)$

9. (i) Prove by induction that, for $n \in \mathbb{Z}^+$

$$\sum_{r=1}^{n} r(r+1)(r+2) = \frac{n(n+1)(n+2)(n+3)}{4}$$

(5)

(ii) Prove by induction that,

$4^n + 6n + 8$ is divisible by 18

for all positive integers n.

(6)

$n=1$ LHS $= 1(2)(3) = 6$ RHS $= \frac{1(2)(3)(4)}{4} = 6$

$\therefore$ true for $n=1$

assume true for $n=k$ $\sum_{1}^{k} r(r+1)(r+2) = \frac{k(k+1)(k+2)(k+3)}{4}$

$n=k+1$ $\sum_{1}^{k+1} r(r+1)(r+2) = \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3)$

$= \frac{k(k+1)(k+2)(k+3) + 4(k+1)(k+2)(k+3)}{4}$

$= \frac{(k+1)(k+2)(k+3)[k+4]}{4}$

$n = k+1 \Rightarrow \frac{n(n+1)(n+2)(n+3)}{4} = \frac{(k+1)(k+2)(k+3)(k+4)}{4}$ $\therefore$ LHS = RHS

$\therefore$ true for $n=1$, true for $n=k+1$ if true for $n=k$ $\therefore$ by Mathematical Induction true for $n \in \mathbb{Z}^+$

ii) $4^n + 6n + 8$

$n=1 \Rightarrow 4^1 + 6(1) + 8 = 18 = 18 \times 1$ $\therefore$ divisible by 18

$\therefore$ true for $n=1$

assume true for $n=k$ $\therefore 4^k + 6k + 8 = 18(n)$ $n \in \mathbb{Z}$

$n=k+1 = 4^{k+1} + 6(k+1) + 8$

$= 4(4^k) + 6k + 14$

$= 4[4^k + 6k + 8] - 18k - 18$

$= 4(18(n)) + 18(-k-1)$

divisible by 18

divisible by 18

$\therefore 4^{k+1} + 6(k+1) + 8$ is divisible by 18

$\therefore$ true for $n=1$ true for $n=k+1$ if true for $n=k$

$\therefore$ by Mathematical induction true for all $n \in \mathbb{Z}^+$